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Diagrammatic Expansion and Metastability in the Random-Field Ising Model

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Within the perturbation diagrammatic expansion we discuss the origin of differences in determinations of the lower critical dimension of the random-field Ising model and show that below four dimensions metastability and hysteresis occur. We also explain the occurrence of a quasicritical $\bar{d}=2$ behavior at weak random fields, which is responsible for local stability of the ordered state above two dimensions.

KEY WORDS: Ising model; random field; diagrammatic expansion; lower critical dimension; metastability; quasicriticality.

1. INTRODUCTION

At the early history of investigation of the critical behavior of the random-field Ising model (RFIM), the perturbation diagrammatic expansion[1-4] seemed to give a proof of a dimensional reduction in critical indices from the d-dimensional RFIM to the $\bar{d}=d-2$ pure Ising model. When applied to determination of the lower critical dimension (LCD) of the RFIM, this reduction gives $d_1=3$. This is, however, in contradiction with the simultaneously developing domain-wall theories,[5-7] which predict $d_1=2$. Since more and more arguments, both theoretical and experimental,[8-13] were presented to support the notion that the LCD is $d_1=2$, the diagrammatic approach was discarded as unreliable, at least below four dimensions. The objections raised against the perturbation theory are essentially of two kinds. First, the Griffiths singularities hinder the summation of diagrams and may screen the most singular contributions to the free energy.[14] Second, because of metastability, the proof of Ref. 4 is

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invalidated.[15] Though not concrete, this criticism reveals two important issues of the RFIM. The first concerns the true critical behavior, the existence of sharp phase transitions, and validity of a dimensional reduction [i.e., either $\bar{d}=d-2$ or $\bar{d}=d-2+\eta(\bar{d})$]. The perturbation theory and scaling-type arguments differ on this significantly in low dimensions ($d<4$). The second issue concerns metastability, irreversibility, and hysteresis effects that affect determination of the LCD. They were completely neglected in the theories based on perturbation expansions.

We discuss in this paper the source of different conclusions on the true critical behavior of the RFIM, which lies in the assumption, based on the linear response theory, of some clustering used in the scaling arguments and violated in the diagrammatic approach. Further, we show, using a simple dimensional analysis, that below four dimensions the random system undergoes a crossover from a quasicritical $\bar{d}=2$ to the true critical $\bar{d}=d-2$ behavior. Because of this, metastability appears and the results of the perturbation theory depend on the way the diagrams are summed. We must investigate separately different histories of the sample. When the sample is cooled in the field (field cooling, FC), long-range order (LRO) cannot be achieved in $d\leqslant 3$, while in the zero-field cooling (ZFC), i.e., when the field is applied only after the sample is cooled below the Curie temperature T_{C0}, LRO persists at weak fields in $d>2$. Thus, LRO is locally stable in $d>2$ and the lower critical dimension is $d_l=2$ even in the expansion into the most divergent diagrams. The criterion for determination of the LCD heretofore used[10] cannot be applied, since it neglects metastability and thus concerns only the existence of a sharp phase transition. The stability of LRO is decided from quasicritical behavior.

2. PERTURBATION THEORY

The model is defined by the lattice Hamiltonian

$$H=-J\sum_{\langle ij\rangle} S_iS_j-\sum_i h_iS_i, \qquad S_i=\pm 1 \tag{1}$$

where h_i is a random field with a site-independent probability distribution $P(h)$. The density of the averaged free energy for the quenched model is

$$f=-\frac{1}{\beta N}\langle \ln \operatorname*{Tr}_S e^{-\beta H}\rangle_{\rm av} \tag{2}$$

where N is the number of lattice sites and $\beta=(k_B T)^{-1}$. Perturbation theory of the critical behavior of the RFIM investigates only the divergent part of the free energy (2). Thus, in the first step, we neglect thermal fluctuations and replace (2) by a mean-field expression:

$$f_{\mathrm{MF}} = -\frac{1}{\beta N}\left\langle J\beta \sum_{\langle ij\rangle} m_i m_j + \beta \sum_i h_i m_i - \sum_i \left[\left(\frac{1+m_i}{2}\right)\ln\left(\frac{1+m_i}{2}\right) + \left(\frac{1-m_i}{2}\right)\ln\left(\frac{1-m_i}{2}\right)\right]\right\rangle_{\mathrm{av}} \tag{3}$$

where m_i is a solution of the inhomogeneous mean-field equations

$$m_i = \tanh \beta\left(h_i + J \sum_{\substack{j \\ |j-i|=1}} m_j\right) \tag{4}$$

In the second step, for small random field h and magnetizations m_i, we replace (4) by the Landau–Ginzburg equation:

$$m_i = \beta h_i(1-\beta^2 h_i^2/3) + \beta J(1-\beta^2 h_i^2) \sum_{\substack{j \\ |i-j|=1}} m_j - \beta^3 J^2 h_i \sum_{\substack{j_1, j_2 \\ |j_\alpha - i|=1}} m_{j_1} m_{j_2} - g_0 \beta^3 J^3 \sum_{\substack{j_1, j_2, j_3 \\ |j_\alpha - i|=1}} m_{j_1} m_{j_2} m_{j_3} \tag{5}$$

Equation (5) is the starting point for the perturbation expansion, that is, expansion in the bare coupling constant $g_0 = 1/3$. The solution of (5) is inserted into (3) and the averaging is done in such a way that only the most divergent contributions are taken into account. Diagrammatically this means that only two types of propagators appear: connected (solid line) and disconnected (crossed line) correlation functions χ and χ^{dis}. They are defined in the momentum space as

$$\chi(q) = \left\langle \frac{\partial m_q}{\partial \beta h_q}\right\rangle_{\mathrm{av}} = \frac{\Sigma(q)}{1 - z\beta J \varepsilon(q)\, \Sigma(q)} \tag{6}$$

$$\chi^{\mathrm{dis}}(q) = \langle m_q m_{-q}\rangle_{\mathrm{av}} = \beta^2 h^2\, \Delta(q)\, \chi(q)^2 \tag{7}$$

where $-1 \leqslant \varepsilon(q) \leqslant 1$ is the dispersion relation on the lattice, z is the coordination number, $h^2 \equiv \langle h_i^2 \rangle_{\mathrm{av}}$, and $\Sigma(q)$ and $\Delta(q)$ are the self-energy and the vertex functions, respectively. The perturbation theory expresses these two functions in diagrams with bare propagators $\chi^0(q)$ and $\chi^{0,\mathrm{dis}}(q)$ for which $\Sigma^0(q) = \Delta^0(q) = 1$. The most divergent part is obtained by neglecting diagrams with loops in χ^0, i.e., if cutting the diagrams along crosses, only connected tree diagrams are left. The functions $\Sigma(q)$ and $\Delta(q)$ are not independent in this theory. They are connected by a functional relation

$$\Delta(q) = \frac{1}{N} \sum_{q'} \frac{\delta \Sigma(q)}{\delta \chi^0(q')} \chi^0(q')^2 + 1 \tag{8}$$

where $\Sigma(q)$ depends on $\chi^0(q')$ and $\chi^{0,\mathrm{dis}}(q')$ through the diagrammatic expansion. Thus, at criticality for $q \to 0$

$$\Sigma(q) \sim \Sigma_0 - \Sigma_1 q^{2-\eta} \tag{9}$$

and

$$\Delta(q) \sim \Delta_0 q^{-\eta} \tag{10}$$

Relation (10) is crucial since it says that $\Delta(q)$ becomes divergent with nonzero η. This divergence breaks down the clustering property of the linear response theory

$$\langle \chi_h(q)\, \chi_h(-q) \rangle_{\mathrm{av}} \xrightarrow[q \to 0]{} \chi(q)\, \chi(-q) \tag{11}$$

where $\chi_h(q)$ is the field-dependent correlation function. Simple global scaling arguments[5–7] always use somehow this property and it is utilized equally in Schwartz's self-consistent theory.[10] It was recently pointed out by Krey[16] that just (11) is the weak point of the theories leading to $d_l = 2$. Up to now, there is no convincing argument whether (10) or (11) holds. It is, of course, beyond perturbation theory to prove (10) or (and) disprove (11). Thus, any kind of rigorous result on this issue would be very helpful. It was shown recently that higher order field cumulants change the powers of the leading divergences,[17] but their effect seems to be the same as that of a very weak random field investigated below. It can also be shown that if (11) is assumed as a constraint in the standard theory,[18] the resulting dimensional reduction is $d \to \bar{d} = d - 2 + \eta(\bar{d})$ as obtained in Refs. 1 and 10. Relation (11) is currently preferred, since it gives the correct LCD. In the next section we show that even (10), and thus the expansion into the most divergent diagrams, does not contradict the local stability of LRO in $d > 2$.

3. DIMENSIONAL ANALYSIS AND THE LOWER CRITICAL DIMENSION

Perturbative investigation of the critical behavior of the RFIM is based on dimensional analysis of diverging diagrams. We shall assume that the only relevant length is the correlation length $\xi \to \infty$ (i.e., one-parameter scaling) and that the critical behavior can be deduced entirely from the dimensionality and the number of components (propagators in closed loops) of the most divergent diagrams. We extract the divergent (dominant at finite ξ) part of the d-dimensional RFIM in a given order of the perturbation expansion (loop expansion) and compare the divergence with the corresponding diagrams of a $\bar{d}$-dimensional pure model in the Landau–Ginzburg form. Neglecting the finite parts, we obtain an order-of-magnitude equivalence and a dimensional reduction $d \to \bar{d}$ of the RFIM to a pure model, not necessarily the Ising model.

The theory is determined by two parameters: random-field amplitude h and temperature T. We shall use dimensionless units, multiples of zJ. We shall investigate only finite temperatures $T < T_{C0}$. First, we assume that h/T does not scale with ξ (i.e., $\xi^{-1} \ll h/T \ll \xi$). We rescale the bare correlation functions

$$\chi^0(q) = \xi^2 \bar{\chi}^0(\xi q), \qquad \chi^{0,\mathrm{dis}}(q) = \xi^4 \bar{\chi}^{0,\mathrm{dis}}(\xi q) \tag{12}$$

where $\bar{\chi}^0$ and $\bar{\chi}^{0,\mathrm{dis}}$ are finite $[\xi^{-1} \ll \bar{\chi}^0(\xi q),\ \bar{\chi}^{0,\mathrm{dis}}(\xi q) \ll \xi]$. The most divergent diagram with L loops, I internal lines, and E external legs scales

$$\Gamma_{I,L}(q_1,..., q_E) = \xi^{L(2-d)+2(E+I)} \bar{\Gamma}_{I,L}(\xi q_1,..., \xi q_E) \tag{13}$$

since just L internal lines are crossed. The corresponding diagram in the $\bar{d}$-dimensional pure model scales

$$\Gamma'_{I,L}(q'_1,..., q'_E) = \xi^{-L\bar{d}+2(E+I)} \bar{\Gamma}'_{I,L}(\xi q'_1,..., \xi q'_E) \tag{14}$$

Comparing the powers of ξ of the both expressions yields $\bar{d} = d - 2$ in all orders of the perturbation expansion.

This argument is valid only if $h/T \gg \xi^{-1}$ or the critical value h_c of the random field at a given temperature is nonzero in all orders of the perturbation expansion. This is violated at weak random fields below four dimensions.[17] The self-consistent one-loop approximation in the lowest order of h^2 gives the most divergent contribution

$$\Sigma = 1 - \frac{(h+m)^2}{T^2} - \frac{h^2}{T^4} \frac{1}{N} \sum_q \frac{\varepsilon(q)^2}{[1 - \Sigma\varepsilon(q)/T]^2} \tag{15}$$

where m is the spontaneous magnetization. The only critical curve in this approximation is $h_c = 0$, $T \leqslant T_{C0}$. The limit $h \to 0$ in the paramagnetic phase ($m = 0$, $T_0 < T_{C0}$) is

$$\langle m_i^2 \rangle_{\mathrm{av}} = \frac{h^2}{T_0^4 N} \sum_q \frac{\varepsilon(q)^2}{[1 - \Sigma\varepsilon(q)/T]^2} \to \lambda_0 \tag{16}$$

where $\lambda_0 = T_0^2(T_{C0} - T_0)$. The h affects the scaling and ξ and h are correlated. From (16) it follows that

$$\xi = (\lambda_0 T_0^2 / f_0)^{1/(4-d)} h^{-\nu_H^0} \tag{17}$$

where f_0 is an integral over the momenta and is of order unity, $\nu_H^0 = 2/(4-d)$. Using relation (17), we obtain a modified scaling of the disconnected correlation function

$$\chi^{0,\mathrm{dis}}(q) = \xi^d \tilde{\chi}^{0,\mathrm{dis}}(\xi q) \tag{18}$$

Neglecting again the finite parts, we obtain $\bar{d} = 2$ for the effective perturbative dimensionality of the RFIM in this region, using (13), (14) and (18). In dimensions $d > 2$, the reduced model has one component corresponding to the internal propagator $\chi^{0,\mathrm{dis}}$ representing fluctuations of the random field. To see what happens when we sum the higher order terms of the perturbation series, we investigate FC and ZFC separately.

We assume in both cases that

$$\xi = f(h, T), \qquad h = g(\xi, T), \qquad T = t(\xi, h) \tag{19}$$

are well-defined functions. Simulating FC, we fix $\xi < \infty$ (but large) and h during the summation of diagrams. For a sufficiently small h and $T \lesssim T_{C0}$, (17) is satisfied in approximation (15). The effective dimensionality $\bar{d} = 2$. Higher order contributions cause an increase in temperature and generate a nonzero critical temperature T_{c2}. The resummation of the whole series modifies a scaling of the correlation length and the disconnected susceptibility as

$$\xi = A(T - T_{c2})^{-\nu_2}, \qquad \chi^{\mathrm{dis}}(q) = \xi^{d-\eta_2} \tilde{\chi}^{\mathrm{dis}}(\xi q) \tag{20}$$

where $\nu_2 \sim 1$, $\eta_2 \sim 0.25$ are critical indices close to those of the pure $\bar{d} = 2$ Ising model. The equality is only approximate, since we disregarded the symmetry factors of the diagrams. According to (17) and (20), the temperature after resummation will be

$$T_1 = T_{c2} + (\lambda_0 T_0^2 / Af)^{1/(d-4)\nu_2} h^{\nu_H^0/\nu_2} \tag{21}$$

Thus, around this temperature, the model effectively behaves like the $d = 2$ pure Ising model. When we lower the temperature in FC, ξ increases and the effective dimension does not change unless

$$T < T_2 \sim T_{c2} + \left(\frac{\lambda_0 T_{c2}^2}{Af} \right)^{1/(d+\eta_2-4)\nu_2} h^{\nu_H^2/\nu_2} \tag{22}$$

where $\nu_H^2 = 2/(4-d-\eta_2)$. Below the temperature T_2, the constraint $\langle m_i^2 \rangle_{\text{av}} \leqslant 1$ interferes. The correlation (17) between h and ξ dies out (h/T no longer scales) and the system undergoes a crossover from the effective dimensionality $\bar{d}=2$ to $\bar{d}=d-2$. The transition at T_{c2} is inevitably destroyed and the true critical behavior (if any) calculated in the effective dimension $\bar{d}=d-2$ slowly settles down.

In ZFC we fix $\xi < \infty$ and $0 < T < T_{C0}$ during the summation. Switching on a small random field of order $h_0 \sim T\xi^{-1/\nu_H^0}$, we are still in the ordered phase in approximation (15). We can bound the dominant contribution of $\chi^{0,\text{dis}}$ by (18) and the system again feels an effective dimension $\bar{d}=2$. The sum of the rest of the most divergent diagrams lowers the starting random field h_0 and generates a critical field $h_{c2} > 0$ below which the state is ordered if $\bar{d} \geqslant 2$. After resummation, the correlation length scales

$$\xi = A'_H (h_{c2} - h)^{-\nu_H^2} \tag{23}$$

If now

$$h < h_{c2}/(1+x), \qquad x = (fA'_H)^{1/2}/m_0 T \tag{24}$$

where m_0 is the zero-field magnetization, the system feels the effective dimension $\bar{d}=2$ and remains ordered. For stronger fields the bound (18) no longer holds, again because of the constraint $\langle m_i^2 \rangle_{\text{av}} \leqslant 1$, and the system undergoes a crossover to the effective dimension $\bar{d}=d-2$. According to this, the transition to the paramagnetic phase cannot be second order if $d \leqslant 3$. In $3 < d < 4$ the transition can, in principle, be second order, which depends on the temperature and the way the crossover occurs.

We now deduce the LCD. We have seen that LRO can exist in ZFC even if there is no sharp transition from the paramagnetic phase. That is, we have shown that LRO is locally stable in $d > 2$. The case $d=2$ must be treated more carefully. From (12) and (18) it follows that $\chi^0(q)$ and $\chi^{0,\text{dis}}(q)$ scale in the same way. Thus, to keep the most divergent terms of the perturbation expansion, we must include the loops in $\chi^0(q)$ (i.e., the thermal fluctuations), since they equally contribute to the most divergent part of the series. Then we have two internal propagators in the loop expansion and the random model reduces to a pure model with $n=2$ components. In such a case, the critical random field $h_{c2}=0$ and LRO at finite temperatures is unstable against the perturbation of a weak random field. In $d > 2$, at weak random fields ordered and disordered phases coexist and we cannot decide, using this qualitative analysis, what phase is at equilibrium. Numerical calculations[19] indicate that it is the ordered one. That is, the lower critical dimension is $d_l = 2$ also in the expansion into the most divergent diagrams.

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